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*Rapport  
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## Modeling of topological derivatives for contact problems

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**Abstract:** The form of topological derivatives for shape functionals for contact problems is established. Methods of numerical evaluation of such derivatives are proposed and compared.

**Key-words:** shape optimization, topological derivative, contact problem, Signorini problem, variational inequality, asymptotic expansion, energy functional.

**Mathematics Subject Classification (2000):** 49Q10, 49Q12, 35J05, 35J50, 35B37

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## Modelisation des dérivées topologiques pour des problèmes de contact.

**Résumé :** La forme de la dérivée topologique est établie pour une classe de problèmes unilatéraux en mécanique du solide. Des méthodes numériques sont proposées et testées pour calculer cette dérivée.

**Mots-clés :** optimisation de formes, dérivée topologique, problème de contact, inégalité variationnelle, analyse asymptotique, fonctionnelle de l'énergie.

# 1 Introduction

We describe briefly the contents of the report. The problem of topology optimisation is considered for free boundary problems of thin obstacle types. The formulae for the first term of asymptotics for energy functionals are derived. The precision of obtained terms is verified numerically. The topological differentiability of solutions to variational inequalities is established. In particular, the so-called *outer asymptotic expansion* for solutions of contact problems in elasticity with respect to singular perturbation of geometrical domain depending on small parameter are derived by an application of nonsmooth analysis. Such results lead to the *topological derivatives* of shape functionals for contact problems. The topological derivatives are used in numerical methods of simultaneous shape and topology optimisation.

In the engineering literature there are many results concerning the shape optimization of contact problems in elasticity. The boundary variations technique for such problems is described in [32] in the framework of nonsmooth analysis combined with the speed method. Nonsmooth analysis is necessary since the shape differentiability of solutions to contact problems is obtained only in the framework of Hadamard differentiability of metric projections onto polyhedral sets in the appropriate Sobolev spaces. However, to our best knowledge, there is no numerical methods for simultaneous shape and topology optimization [38] of contact problems. The main difficulty in analysis of contact problems is associated with the nonlinearity of the *nonpenetration condition* over the contact zone which makes the boundary value problem nonsmooth. In the paper we propose a method for numerical evaluation of topological derivatives for such problems. The notion of topological derivative of a shape functional is introduced in [33], [34]. The knowledge of topological derivatives is required for the optimality conditions of simultaneous shape and topology optimisation. The topological derivative of a given shape functional can be determined from the variations of the shape functional created by the variations of the topology of geometrical domains. The topology variations are defined by nucleation of small holes or cavities or more generally small defects in geometrical domains. The modern mathematical background for evaluation of such derivatives by the asymptotic analysis techniques of boundary value problems is established in [27]. In [27] the error estimates for asymptotic approximations of solutions to boundary value problems in singularly perturbed geometrical domains are provided in the weighted Hölder spaces. The asymptotic approximations of solutions are used in order to establish the explicit formulae for the topological derivatives of shape functionals.

The main idea we use to derive the topological derivatives for contact problems is the modification of the energy functional by an appropriate correction term and subsequent minimisation of the resulting energy functional over the cone of admissible displacements. Such an approach leads to the *outer* approximations of solutions to variational inequalities. The following step in asymptotic approximation of solutions is an application of self adjoint extensions of elliptic operators [28] with asymptotic point conditions for the contact problems which results in the asymptotically exact approximations. We point out that in the framework of self adjoint extensions studied in [28] for linear problems, the approximate solutions with asymptotic point conditions at the centre of the hole  $\mathcal{O}$  are critical points of the appropriate energy functional. We restrict ourselves to the case of outer approxima-

tions of solutions. Such an approach is justified, by the applications to numerical methods of topology optimisation. For linear problems, outer approximations are used e.g., in [8], see also [1], for derivation of topological derivatives for isotropic elasticity. However, the complete asymptotic analysis necessary to justify the derivation of topological derivatives for general linear boundary value problems is performed in [27].

In the paper we derive useful formulae for the correction terms of the energy functionals. We restrict ourselves to two dimensional problems and to singular perturbations of geometrical domains in the form of small discs. Two representative problems are considered including the Signorini problem and the frictionless contact problem in linear elasticity.

The correction terms are derived in such a form, that the numerical verification of its precision is straightforward. On the other hand, the terms are directly used to establish the topological differentiability of solutions to variational inequalities. As a result, the one term *outer* expansion of solutions is derived for a class of nonlinear problems. Outer expansion means that the expansion is precise far from the hole, the expansion precise near the hole is called *inner* expansion and usually the matching procedure is applied [9], [20] to construct the global asymptotic approximation of solutions to boundary problems in singularly perturbed geometrical domains.

The topological differentiability of the energy functional for scalar Signorini problem is obtained in [5] under some regularity assumptions. We obtain the results on the topological differentiability of solutions to Signorini problem and to the contact problem in elasticity in the framework of nonsmooth analysis. It means that we establish the first order outer asymptotic approximation of the solutions to the variational inequality in the singularly perturbed geometrical domains. Such an approximation is constructed by replacing the exact energy functional by its asymptotic approximation and by subsequent minimization of the approximate energy over the convex cone. Equivalently, it means that the approximate bilinear form is constructed and the variational inequality is considered for such a form. Since the singular perturbation of geometrical domain is replaced by the regular perturbation of the bilinear form, we can apply the well established technique of conical differentiability of solutions to variational inequalities to derive the required outer approximation of solutions. Actually, our approach consists of two steps. First, the energy functional is analysed in the domains including small holes. The first term of asymptotics is identified for the energy functional, and the form of the term is selected in such a way that the numerical computations are easy to perform. We provide results of computations in order to compare the different equivalent forms of the correction terms of the energy functionals. In the second step, for the approximate of energy functional, the minimization problem over a convex cone in the energy space is considered. This problem leads to variational inequality depending on the small parameter  $\rho$ , which admits the unique solution for the parameter  $\rho \in (0, \rho_{\min}]$ . The parameter  $\rho$  describes the size of the opening  $B(\rho)$  in the domain  $\Omega_\rho = \Omega \setminus \overline{B(\rho)}$ , and  $\rho_{\min}$  is sufficiently small to assure the existence of an unique solution for the variational inequality. The new result of the paper is the proof, that the solutions of variational inequalities are Hadamard differentiable with respect to  $\rho$ , at  $\rho = 0+$ . The conical differential of solution to variational inequality is given by a unique solution of an auxiliary variational inequality.

The auxiliary variational inequality is explicitly determined. In particular, such a result on conical differentiability can be used for simultaneous shape and topology optimization for contact problems in the way proposed in [38].

The procedure we use in the paper can be described in the following way.

First, let us consider the Signorini problem. It is a classical free boundary problem with an obstacle on the boundary  $\Gamma_s$ . The solution  $u(\Omega)$  of such a problem with the domain of integration  $\Omega \subset \mathbb{R}^2$  satisfies the variational inequality

$$u = u(\Omega) \in K : \int_{\Omega} \nabla u \cdot \nabla (v - u) \geq 0 \quad \forall v \in K \quad (1)$$

where

$$K(\Omega) = \{v \in H^1(\Omega) | v = g \text{ on } \Gamma_0, v \geq 0 \text{ on } \Gamma_s\}. \quad (2)$$

For the domain  $\Omega_\rho$ , with the small hole  $B(\rho)$  in the form of a disc  $B(\rho) = \{x : |x - \mathcal{O}| < \rho\} \subset \Omega$ ,  $\mathcal{O}$  is the centre of the hole, the solution of the Signorini problem is denoted by  $u = u(\Omega_\rho)$ , such a solution is unique for small  $\rho$ . We assume in the sequel that  $\mathcal{O}$  is the origin. In addition,  $u(\Omega_\rho)$  satisfies the homogeneous Neumann condition on the boundary  $\Gamma_\rho$  of the hole  $B(\rho)$ .

We are interested in the asymptotic behaviour of  $u(\Omega_\rho) \in H^1(\Omega_\rho)$  for  $\rho \rightarrow 0+$ . The energy functional for  $\Omega_\rho$ ,  $\rho \geq 0$ ,  $\rho$  small enough,

$$\mathcal{E}(\Omega_\rho) = \frac{1}{2} \int_{\Omega_\rho} \|\nabla u(\Omega_\rho)\|^2$$

admits the asymptotic expansion [[5], p. 755, (4.28)]

$$\mathcal{E}(\Omega_\rho) = \mathcal{E}(\Omega) + \frac{\rho^2 \pi}{2} |\nabla u(\Omega; \mathcal{O})|^2 + O(\rho^{3-\delta})$$

for some  $0 < \delta < 1$ . The first order correction term  $\rho^2 b(u(\Omega), u(\Omega)) = -\rho^2 \pi |\nabla u(\Omega; \mathcal{O})|^2$  in the energy functional can be represented in the equivalent form of a line integral over a circle  $\Gamma_R = \{x : |x - \mathcal{O}| = R\}$  with the centre at  $\mathcal{O}$

$$b(u, u) = -\frac{1}{2\pi R^6} \left[ \left( \int_{\Gamma_R} u x_1 ds \right)^2 + \left( \int_{\Gamma_R} u x_2 ds \right)^2 \right]$$

Therefore, we can define the new energy functional defined on  $\Omega$  and depending on the small parameter  $\rho$ ,

$$E_0(\rho; w) = \frac{1}{2} \int_{\Omega} \|\nabla w\|^2 dx + \rho^2 b(w, w)$$

The functional  $E_0(\rho; w)$  substantially differs from the energy defined for selfadjoint extensions of elliptic operators, which includes the so-called separated asymptotics [24], [28], and

leads to asymptotically exact approximation of  $u(\Omega_\rho)$  in the linear case. Minimizing the functional  $E_0(\rho; w)$  over the set  $K = K(\Omega)$  leads to the solution  $u_\rho$  which is an *outer* approximation of the solution  $u(\Omega_\rho)$ . For the outer approximation we have the following expansion in  $H^1(\Omega)$ , with respect to the small parameter,

$$u_\rho = u(\Omega) + \rho^2 q + o(\rho^2) . \quad (3)$$

Expansion (3) follows by an application of the conical differentiability of solutions to variational inequalities.

**Remark 1** *From the results of ([20], Chapters 2 and 4) it follows that we have the following properties of the asymptotic approximations  $U$  for the problem in a ball. For the equation*

$$-\Delta u = 0 \quad \text{in } B_1(\mathcal{O}) \setminus \overline{\omega_\varepsilon}, \quad u = g \quad \text{on } S_1(\mathcal{O}), \quad \partial_n u = 0 \quad \text{on } \partial\omega_\varepsilon$$

*The following estimate hold true*

$$\|u - U; H^{l+1/2}(S_1(\mathcal{O}))\| + \|\partial_n(u - U); H^{l-1/2}(S_1(\mathcal{O}))\| \leq c_\kappa \varepsilon^{1-\kappa} \|g; H^{l+1/2}(S_1(\mathcal{O}))\| ,$$

*where  $\kappa > 0$ ,  $c_\kappa \rightarrow \infty$  as  $\kappa \rightarrow 0+$ ;  $c_\kappa$  depends only on  $l$  and  $\omega$ ;  $l \in \mathbb{N} = \{1, 2, \dots\}$  can be taken arbitrary.*

*Here  $U$  stands for the asymptotic approximation of  $u$ , and  $\omega$  is a hole,  $\omega_\varepsilon = \varepsilon\omega$ , the origin  $\mathcal{O} \in \omega$ .*

*The same estimate holds [25] with  $l + 1/2$  replaced by  $l + \alpha$  for any  $\alpha \in (0, 1)$ .*

Now, let us consider the asymptotic analysis of solutions to variational inequalities in the abstract setting.

We consider the variational inequality

$$u_\rho \in K : \quad a(\rho; u_\rho, v - u_\rho) \geq 0 \quad \forall v \in K ,$$

where the bilinear form is given by

$$a(\rho; u, v) = a(0; u, v) + \rho^2 b(u, v)$$

and  $\rho > 0$  is a small parameter. The bilinear form  $a(\rho; u, v)$  is determined by asymptotic analysis of the appropriate energy functional depending on the problem. The parameter  $\rho$  measures the size of small defect in the geometrical domain of integration  $\Omega_\rho$  for solutions of the variational inequality. In particular, we show that the perturbation  $b(u, v)$  can be expressed in two different ways.

- as the pointwise values of some differential expression;
- or equivalently, as a line integral over a circle in the domain of integration.



Numerical examples confirm that the second expression is more suitable for the numerical methods.

The paper contains two parts. In the second part the formulae are derived for the perturbation  $b(u, v)$  for two different two dimensional problems by an application of the asymptotic analysis.

In the first part of the paper, for the variational inequality depending on the small parameter  $\rho > 0$ , the expansion of solutions is derived for  $\rho > 0$  sufficiently small

$$u_\rho = u + \rho^2 q + o(\rho^2) .$$

The term  $q$  is called the *topological derivative* of the solution  $u(\Omega)$  to the variational inequality. In fact, our construction results in the first term of asymptotics of solutions to variational inequalities, which is asymptotically exact far from the geometrical singularity. However, we do not require any additional regularity on the unknown solutions in order to derive the above expansion. Such a regularity is usually required in the existing literature, and in general cannot be verified. We refer the reader to [2], [3], [4], [5], [19], [20], [21], [22], [23] for the related results on asymptotic analysis of energy functionals and of variational inequalities. Self adjoint extensions of elliptic operators are analysed in [7], [14], [29], for problems with geometrical singularities. Applications of such asymptotic analysis to shape optimization are given in [27], [28], [34], [38], [8].

## 1.1 Signorini problem

We establish the conical differentiability of solutions to Signorini problem with respect to the small parameter. The obtained expansion of solutions to the Signorini problem can be interpreted as the first order *outer asymptotic expansion* in the spirit e.g., of [27]. So, this way, we can define the topological derivatives of some shape functionals, including the energy functional, for variational inequalities. To our best knowledge this is the first result in this direction derived without any assumptions on the strict complementarity conditions for the unknown solutions to the obstacle problems.

In order to introduce the Signorini problem in  $\Omega_\rho$  we need the bilinear form  $a(\cdot, \cdot)$  defined over the domain of integration  $\Omega_\rho$ , the convex set  $K(\Omega_\rho) \subset H^1(\Omega_\rho)$ , the linear form  $L(\cdot)$  which is assumed to be zero in what follows. We are going to define the variational inequality over the space  $H^1(\Omega_\rho)$ , where  $\Omega_\rho = \Omega \setminus B(\rho)$ , with the small ball  $B(\rho) = \{x \in \mathbb{R}^2 \mid \|x - \mathcal{O}\| < \rho\}$  excluded from  $\Omega$ , here the origin  $\mathcal{O}$  is the centre of the ball. So, the variational inequality reads :

Find  $u(\Omega_\rho) \in K(\Omega_\rho) = \{v \in H^1(\Omega_\rho) \mid v = g \text{ on } \Gamma_0, v \geq 0 \text{ on } \Gamma_s\}$  such that  $a(u, v - u) \geq L(v - u)$  for all  $v \in K(\Omega_\rho)$ .

The unique solution  $u(\Omega_\rho)$  of the variational inequality depends on the small parameter  $\rho$ . In order to analyse the dependence of the solution  $u(\Omega_\rho)$  on  $\rho$  we proceed in the following way. The bilinear form

$$a(u, v) = \int_{\Omega_\rho} \nabla u \cdot \nabla v dx$$

defined on the variable geometrical domain  $\Omega_\rho$  is linearized using the asymptotic expansions technique for the energy functionals. The linearized bilinear form is denoted by  $a(\rho; \cdot, \cdot)$  and contains in two dimensions the term of order zero with respect to  $\rho$  and the first order perturbation depending on  $\rho^2$ . In Section 2.1 the following bilinear form is identified as the first order perturbation

$$b(w, v) = -\frac{1}{\pi R^6} \int_{\Gamma_R} w x_1 ds \int_{\Gamma_R} v x_1 ds - \frac{1}{\pi R^6} \int_{\Gamma_R} w x_2 ds \int_{\Gamma_R} v x_2 ds .$$

The bilinear form  $b(w, v)$  is continuous on the space  $H^1(\Omega)$ , where  $\Gamma_R = \partial B(R)$ , and the ball  $B(R) \subset \Omega$ . Thus, the bilinear form  $a(\cdot, \cdot)$  is replaced in the variational inequality over  $\Omega$  by the integral expression, depending only on  $\rho$  as a small parameter,

$$a(\rho; w, v) = \int_{\Omega} \nabla w \cdot \nabla v dx + \rho^2 b(w, v) , \quad (4)$$

$a(\rho; w, v)$  is defined over the domain of integration  $\Omega \subset \mathbb{R}^2$  with the smooth boundary  $\partial\Omega = \Gamma_0 \cup \Gamma_s$ , with  $w, v$  in the energy space  $H^1(\Omega)$ . We denote

$$H_{\Gamma_0}^1(\Omega) = \{v \in H^1(\Omega) | v = 0 \text{ on } \Gamma_0\}$$

and note that  $\{K(\Omega) - K(\Omega)\} \subset H_{\Gamma_0}^1(\Omega)$ , however  $K(\Omega) \subset H^1(\Omega)$ , nevertheless the tangent cone  $T_K(v) \subset H_{\Gamma_0}^1(\Omega)$  for all  $v \in K(\Omega)$ .

Thus, the singular perturbation of the geometrical domain  $\Omega_\rho$  is replaced in the variational inequality under considerations by the *regular* perturbation of the bilinear form. For such regular perturbation the standard sensitivity analysis of variational inequalities over polyhedral sets in Dirichlet spaces applies (see e.g., [31], [12]) and the first order expansion of the solution with respect to the small parameter is obtained. We refer the reader to [5] for the direct asymptotic analysis of the energy functional for the Signorini problem.

We show that for  $\rho$  small enough, bilinear form (4) is elliptique (coercive) on the space  $H_{\Gamma_0}^1(\Omega)$ . To this end we observe that

$$\begin{aligned} \frac{\rho^2}{\pi R^6} \left[ \left( \int_{\Gamma_R} w x_1 ds \right)^2 + \left( \int_{\Gamma_R} w x_2 ds \right)^2 \right] &\leq \frac{\rho^2}{\pi R^6} \int_{\Gamma_R} w^2 ds \int_{\Gamma_R} (x_1^2 + x_2^2) ds = \\ &= \frac{2\rho^2}{R^3} \int_{\Gamma_R} w^2 ds \leq C(\Omega) \frac{2\rho^2}{R^3} \int_{\Omega} \|\nabla w\|^2 dx \end{aligned} \quad (5)$$

hence

$$a(\rho; w, w) \geq \left( 1 - C(\Omega) \frac{2\rho^2}{R^3} \right) \int_{\Omega} \|\nabla w\|^2 dx \geq \frac{1}{2} \int_{\Omega} \|\nabla w\|^2 dx$$

for  $\rho \in \left[ 0, \frac{1}{2} \sqrt{\frac{R^3}{C(\Omega)}} \right]$ , which completes the proof of the coercitivity. Therefore, for  $\rho$  sufficiently small we can consider the variational inequality with the bilinear form  $a(\rho; w, v)$  over

the convex set (2).

Let  $u_\rho$  denotes the unique solution of the following variational inequality

$$u_\rho \in K : \quad a(\rho; w, v - w) \geq 0 \quad \forall v \in K . \quad (6)$$

**Theorem 1** *For  $\rho$  sufficiently small we have the following expansion of the solution  $u_\rho$ , with respect to the parameter  $\rho$ , at  $0+$ ,*

$$u_\rho = u + \rho^2 q + o(\rho^2) , \quad (7)$$

where the outer topological derivative  $q$  of the solution  $u = u(\Omega)$  to the Signorini problem is given by the unique solution of the following variational inequality

$$q \in \mathcal{S}_K(u) = \{v \in H_{\Gamma_0}^1(\Omega) | v \geq 0 \text{ on } \Xi(u) , \quad a(u, v) = 0\} \quad (8)$$

$$a(q, v - q) + b(u, v - q) \geq 0 \quad \forall v \in \mathcal{S}_K(u) . \quad (9)$$

The coincidence set  $\Xi(u) = \{x \in \Gamma_s | u(x) = 0\}$  is well defined [31] for any function  $u \in H^1(\Omega)$ , and  $u \in K \subset V = H^1(\Omega)$  is the solution of the variational inequality for  $\rho = 0$ .

**Proof.** The result follows by an abstract result on the differentiability of metric projection onto the polyhedric convex set in Dirichlet space. To apply the result for the Signorini problem we have to verify that the set  $K$  satisfy property (12) given below. First, we introduce the notation. Given  $u_0 \in K$  we define

$$C_K(u_0) = \{v \in V; \exists t > 0 \text{ such that } u_0 + tv \in K\} , \quad (10)$$

$$T_K(u_0) = \overline{C_K(u_0)} . \quad (11)$$

$T_K(u_0)$  is called the tangent cone to  $K \subset V$  at  $u_0 \in K$ . For the convex sets with unilateral constraints in function spaces, the tangent cones are determined in [31]. The space  $H^1(\Omega)$  is a Dirichlet space, and the so-called capacity can be introduced in such a space. However, we are going to define all necessary objects without referring to the capacity of sets. For any positive measure  $\mu$  of finite energy [31] living on the coincidence set  $\Xi = \{x \in \Gamma_s | u_0(x) = 0\}$  we introduce the cone

$$\mathfrak{D}(\mu) = \mu^\perp = \{v \in H_{\Gamma_0}^1(\Omega) | \langle \mu, v \rangle = \mu[v] = \int v d\mu = 0\}$$

It is shown in [31] that in the Dirichlet space, which applies to the Sobolev space  $H_{\Gamma_0}^1(\Omega)$ , the following condition is satisfied for all  $u_0 \in K$  and all positive measures  $\mu$  of finite energy living on  $\Xi$ ,

$$T_K(u_0) \cap \mathfrak{D}(\mu) = \overline{C_K(u_0) \cap \mathfrak{D}(\mu)} . \quad (12)$$

We denote  $u_0 = u(\Omega)$  and define the positive measure  $\mu$  of finite energy

$$\langle \mu, v \rangle = \int_{\Omega} \nabla u_0 \cdot \nabla v dx .$$

Thus, applying the results of [31] we have

$$\mathcal{S}_K(u_0) = T_K(u_0) \cap \mathfrak{D}(\mu) = \{v \in H_{\Gamma_0}^1(\Omega) | v \geq 0 \text{ on } \Xi(u), \quad a(u_0, v) = 0\}$$

Since the convex set  $K \subset H^1(\Omega)$  is polyhedric, the Hadamard differential of metric projection at  $u_0 \in K$  in the space  $H^1(\Omega)$  is given by the metric projection onto the cone  $\mathcal{S}_K(u_0)$ . We can use an abstract result given by Theorem 2 proved in [[32], Theorem 4.14, Section 4.2] for the sensitivity analysis of solutions to variational inequalities. Theorem 2 is used for (6) with  $t = \rho^2$  and  $a_t(\cdot, \cdot) = a(\rho; \cdot, \cdot)$ . The mapping  $\Pi$  in (16) is conically differentiable, with the differential  $\Pi'h$  given by the unique solution to the variational inequality

$$\Pi'h \in \mathcal{S}_K(u_0) : \quad a(\Pi'h, v - \Pi'h) \geq \langle h, v - \Pi'h \rangle \quad \forall v \in \mathcal{S}_K(u_0),$$

and with  $\langle \mathcal{A}'w, v \rangle = b(w, v)$  for all  $w, v \in V$ . Furthermore, we have  $f_t = 0$ . Hence all assumptions of Theorem 2 are verified and by (18) with (19), for the variational inequality (6), the expansion (7) follows. This completes the proof.

For the convenience of the reader we recall here the abstract result which is a generalization of the implicate function theorem for variational inequalities.

Let  $K \subset V$  be a convex and closed subset of a Hilbert space  $V$ , and let  $\langle \cdot, \cdot \rangle$  denote the duality pairing between  $V'$  and  $V$ , where  $V'$  denotes the dual of  $V$ .

We shall consider the following family of variational inequalities depending on a parameter  $t \in [0, \delta)$ ,  $\delta > 0$ ,

$$y_t \in K : \quad a_t(y_t, \varphi - y_t) \geq \langle f_t, \varphi - y_t \rangle \quad \forall \varphi \in K. \quad (13)$$

Moreover, let  $y_t = \mathcal{P}_t(f_t)$  be a solution to (13). Let us note, that for  $f_t = 0$  and  $y_t = \mathcal{P}_t(0)$  we obtain  $y' = \Pi'(-\mathcal{A}'y_0)$  in (19) which is the case for (6).

**Theorem 2** *Let us assume that*

- *the bilinear form  $a_t(\cdot, \cdot) : V \times V \rightarrow \mathbb{R}$  is coercive and continuous uniformly with respect to  $t \in [0, \delta)$ . Let  $\mathcal{A}_t \in \mathcal{L}(V; V')$  be the linear operator defined as follows  $a_t(\phi, \varphi) = \langle \mathcal{A}_t \phi, \varphi \rangle \quad \forall \phi, \varphi \in V$ ; it is supposed that there exists  $\mathcal{A}' \in \mathcal{L}(V; V')$  such that*

$$\mathcal{A}_t = \mathcal{A}_0 + t\mathcal{A}' + o(t) \quad \text{in } \mathcal{L}(V; V'). \quad (14)$$

- *for  $t > 0$ ,  $t$  small enough, the following equality holds*

$$f_t = f_0 + tf' + o(t) \quad \text{in } V', \quad (15)$$

*where  $f_t, f_0, f' \in V'$*

- *$K \subset V$  is convex and closed, and for the solutions to the variational inequality*

$$\Pi f = \mathcal{P}_0(f) \in K : \quad a_0(\Pi f, \varphi - \Pi f) \geq \langle f, \varphi - \Pi f \rangle \quad \forall \varphi \in K \quad (16)$$

the following differential stability result holds

$$\forall h \in V' : \quad \Pi(f_0 + \varepsilon h) = \Pi f_0 + \varepsilon \Pi' h + o(\varepsilon) \quad \text{in } V \quad (17)$$

for  $\varepsilon > 0$ ,  $\varepsilon$  small enough, where the mapping  $\Pi' : V' \rightarrow V$  is continuous and positively homogeneous and  $o(\varepsilon)$  is uniform, with respect to  $h \in V'$ , on compact subsets of  $V'$ .

Then the solutions to the variational inequality (13) are right-differentiable with respect to  $t$  at  $t = 0$ , i.e. for  $t > 0$ ,  $t$  small enough,

$$y_t = y_0 + ty' + o(t) \quad \text{in } V, \quad (18)$$

where

$$y' = \Pi'(f' - A'y_0) . \quad (19)$$

**Remark 2** Let us observe, that the first order correction of energy functional, and therefore of the bilinear form is given by equivalent expression (65),

$$a(\rho; v, v) = \int_{\Omega} \|\nabla v\|^2 dx - \pi \rho^2 e_v(\mathcal{O})$$

with the energy density  $e_v(\mathcal{O}) = |\nabla v(\mathcal{O})|^2$  evaluated at the origin.

## 1.2 Contact problem in elasticity

We establish the same result on the conical differentiability of solutions for two dimensional contact problem in the elasticity. We consider the bounded domain  $\Omega$  with the boundary  $\partial\Omega = \Gamma_0 \cup \Gamma_c$ . On  $\Gamma_0$  the displacement vector of the elastic body is given, on  $\Gamma_c$  the frictionless contact conditions are prescribed. To specify the weak formulation we need an expression for the symmetric bilinear form and for the convex set  $K \subset H^1(\Omega)^2$ .

The method of analysis is the same as in the case of Signorini problem. We start with the formulation of the free boundary problem in unperturbed domain  $\Omega$ . The form of variational inequality is straightforward.

### 1) Contact problem in $\Omega$

Find  $\mathbf{u} = \mathbf{u}(\Omega) = (u_1, u_2)$  and  $\sigma = (\sigma)_{ij}$ ,  $i, j = 1, 2$ , such that

$$-\mathbf{div} \sigma = \mathbf{f} \quad \text{in } \Omega , \quad (20)$$

$$C\sigma - \epsilon(\mathbf{u}) = 0 \quad \text{in } \Omega , \quad (21)$$

$$\mathbf{u} = 0 \quad \text{on } \Gamma_0 , \quad (22)$$

$$\mathbf{u}\nu \geq 0, \quad \sigma_\nu \leq 0, \quad \sigma_\nu \mathbf{u}\nu = 0 \quad \sigma_\tau = 0 \quad \text{on } \Gamma_c . \quad (23)$$

Here

$$\sigma_\nu = \sigma_{ij}\nu_j\nu_i, \quad \sigma_\tau = \sigma_\nu - \sigma_\nu = \{\sigma_\tau^i\}_{i=1}^2, \quad \sigma\nu = \{\sigma_{ij}\nu_j\}_{i=1}^2,$$

$$\epsilon_{ij}(\mathbf{u}) = \frac{1}{2}(u_{i,j} + u_{j,i}), \quad i, j = 1, 2, \quad \epsilon(\mathbf{u}) = (\epsilon_{ij})_{i,j=1}^2,$$

$$\{C\sigma\}_{ij} = c_{ijkl}\sigma_{kl}, \quad c_{ijkl} = c_{jikl} = c_{klji}, \quad c_{ijkl} \in L^\infty(\Omega).$$

The Hooke's tensor  $C$  satisfies the ellipticity condition

$$c_{ijkl}\xi_{ji}\xi_{kl} \geq c_0|\xi|^2, \quad \forall \xi_{ji} = \xi_{ij}, \quad c_0 > 0, \quad (24)$$

and we have used the summation convention over repeated indices.

When the topology of  $\Omega$  is changed, we have the following contact problem in the domain  $\Omega_\rho$  with the small hole  $B(\rho)$ .

### 2) Contact problem in $\Omega_\rho$

Find  $\mathbf{u} = \mathbf{u}(\Omega_\rho) = (u_1, u_2)$  and  $\sigma = (\sigma)_{ij}$ ,  $i, j = 1, 2$ , such that

$$-\mathbf{div} \sigma = \mathbf{f} \quad \text{in } \Omega_\rho, \quad (25)$$

$$C\sigma - \epsilon(\mathbf{u}) = 0 \quad \text{in } \Omega_\rho, \quad (26)$$

$$\mathbf{u} = 0 \quad \text{on } \Gamma_0, \quad (27)$$

$$\sigma\nu = 0 \quad \text{on } \Gamma_\rho, \quad (28)$$

$$\mathbf{u}\nu \geq 0, \quad \sigma_\nu \leq 0, \quad \sigma_\nu \mathbf{u}\nu = 0 \quad \sigma_\tau = 0 \quad \text{on } \Gamma_c. \quad (29)$$

We assume for simplicity that the case of isotropic elasticity is considered, thus the symmetric bilinear form associated with the boundary value problem (20)-(23) is given by

$$a(\mathbf{u}, \mathbf{u}) = \int_\Omega [(\lambda + \mu)(\epsilon_{11} + \epsilon_{22})^2 + \mu(\epsilon_{11} - \epsilon_{22})^2 + \mu\gamma_{12}^2], \quad (30)$$

where the notation for isotropic elasticity is fixed in Section 3.

The problem (25)-(29) is approximated by the problem with modified bilinear form in the following way.

### 3) Approximation of contact problem in $\Omega_\rho$

We determine the modified bilinear form as a sum of two terms, as it is for the energy functional, the first term defines the elastic energy in the domain  $\Omega$ , the second term is a correction term, determined in Section 3.4 by formula (101). The correction term is quite complicated to evaluate, and we do not provide its explicit form, such a form is actually defined by the formulae in Section 3. The values of the symmetric bilinear form  $a(\rho; \cdot, \cdot)$  are given by the expression

$$a(\rho; \mathbf{v}, \mathbf{v}) = a(\mathbf{u}, \mathbf{u}) + \rho^2 b(\mathbf{v}, \mathbf{v}). \quad (31)$$

The derivative  $b(\mathbf{v}, \mathbf{v})$  of the bilinear form  $a(\rho; \mathbf{v}, \mathbf{v})$  with respect to  $\rho^2$  at  $\rho = 0+$  is given by the expression

$$b(\mathbf{v}, \mathbf{v}) = -2\pi e_{\mathbf{v}}(0) - \frac{2\pi\mu}{\lambda + 3\mu}(\sigma_{11}\delta_1 - \sigma_{12}\delta_2), \quad (32)$$

where all the quantities are evaluated for the displacement field  $\mathbf{v}$  according to formulae (73), (74), (76), (101), (92), we provide the line integrals which defines all terms in (32) below.

Hence, we can determine the bilinear form  $a(\rho; \mathbf{v}, \mathbf{w})$  for all  $\mathbf{v}, \mathbf{w}$ , from the equality

$$2a(\rho; \mathbf{v}, \mathbf{w}) = a(\rho; \mathbf{v} + \mathbf{w}, \mathbf{v} + \mathbf{w}) - a(\rho; \mathbf{w}, \mathbf{w}).$$

In the same way the bilinear form  $b(\mathbf{v}, \mathbf{w})$  is determined from the formula for  $b(\mathbf{v}, \mathbf{v})$ . The convex set is defined in this case by

$$\mathbf{K} = \{\mathbf{v} \in H^1(\Omega)^2 \mid \mathbf{v}\nu \geq 0 \text{ on } \Gamma_c, \quad \mathbf{v} = \mathbf{g} \text{ on } \Gamma_0\}. \quad (33)$$

Let us consider the following variational inequality which provides a sufficiently precise for our purposes approximation  $\mathbf{u}_\rho$  of the solution  $\mathbf{u}(\Omega_\rho)$  to contact problem (25)-(29),

$$\mathbf{u}_\rho \in \mathbf{K} : \quad a(\rho; \mathbf{u}, \mathbf{v} - \mathbf{u}) \geq L(\rho; \mathbf{v} - \mathbf{u}) \quad \forall \mathbf{v} \in \mathbf{K}. \quad (34)$$

The result obtained is the following, for simplicity we assume that the linear form  $L(\rho; \cdot)$  is independent of  $\rho$ .

**Theorem 3** *For  $\rho$  sufficiently small we have the following expansion of the solution  $u_\rho$  with respect to the parameter  $\rho$  at  $0+$ ,*

$$\mathbf{u}_\rho = \mathbf{u}(\Omega) + \rho^2 \mathbf{q} + o(\rho^2) \quad \text{in } H^1(\Omega)^2, \quad (35)$$

where the topological derivative  $\mathbf{q}$  of the solution  $\mathbf{u}(\Omega)$  to the contact problem is given by the unique solution of the following variational inequality

$$\mathbf{q} \in \mathcal{S}_{\mathbf{K}}(\mathbf{u}) = \{\mathbf{v} \in (H_{\Gamma_0}^1(\Omega))^2 \mid \mathbf{v}\nu \leq 0 \text{ on } \Xi(\mathbf{u}), \quad a(0; \mathbf{u}, \mathbf{v}) = 0\} \quad (36)$$

$$a(0; \mathbf{q}, \mathbf{v} - \mathbf{q}) + b(\mathbf{u}, \mathbf{v} - \mathbf{q}) \geq 0 \quad \forall \mathbf{v} \in \mathcal{S}_{\mathbf{K}}(\mathbf{u}). \quad (37)$$

The coincidence set  $\Xi(\mathbf{u}) = \{x \in \Gamma_s \mid \mathbf{u}(x) \cdot \nu(x) = 0\}$  is well defined [31] for any function  $\mathbf{u} \in H^1(\Omega)^2$ , and  $\mathbf{u} \in \mathbf{K}$  is the solution of variational inequality (33) for  $\rho = 0$ .

The proof of Theorem 3 is similar to the proof of Theorem 1, and therefore is omitted.

**Remark 3** *In the linear case, it can be shown that  $\|\mathbf{u}(\Omega_\rho) - \mathbf{u}_\rho\| = o(\rho^2)$  in the norm of appropriate weighted space. We refer the reader to [27] for the related error estimates in the Hölder weighted spaces. In general, we cannot expect that  $\mathbf{u}_\rho$  is close to  $\mathbf{u}(\Omega_\rho)$  in the vicinity of  $B_\rho$ , therefore the weighted spaces should be used for error estimates.*

For the convenience of the reader we provide the explicit formulae for the terms in  $b(\mathbf{v}, \mathbf{v})$  defined by (32), we refer to section 3.3 for details. We have

$$\begin{aligned} 2\pi e_{\mathbf{v}}(0) &= \frac{\pi(\lambda + \mu)}{\pi^2 R^6} \left( \int_{\Gamma_R} (v_1 x_1 + v_2 x_2) ds \right)^2 + \\ &+ \frac{\mu}{\pi^2 R^6} \left( \int_{\Gamma_R} \left( (1 - 9k)(v_1 x_1 - v_2 x_2) + \frac{12k}{R^2}(v_1 x_1^3 - v_2 x_2^3) \right) ds \right)^2 + \\ &+ \frac{\mu}{\pi^2 R^6} \left( \int_{\Gamma_R} \left[ (1 + 9k)(v_1 x_2 + v_2 x_1) - \frac{12k}{R^2}(v_1 x_2^3 + v_2 x_1^3) \right] ds \right)^2, \end{aligned} \quad (38)$$

with

$$\begin{aligned}\sigma_{II} &= \frac{\mu}{\pi R^3} \int_{\Gamma_R} \left[ (1 - 9k)(v_1 x_1 - v_2 x_2) + \frac{12k}{R^2}(v_1 x_1^3 - v_2 x_2^3) \right] ds, \\ \sigma_{12} &= \frac{\mu}{\pi R^3} \int_{\Gamma_R} \left[ (1 + 9k)(v_1 x_2 + v_2 x_1) - \frac{12k}{R^2}(v_1 x_2^3 + v_2 x_1^3) \right] ds,\end{aligned}$$

and

$$\begin{aligned}\delta_1 &= \frac{9k}{\pi R^3} \int_{\Gamma_R} \left[ (v_1 x_1 - v_2 x_2) - \frac{4}{3R^2}(v_1 x_1^3 - v_2 x_2^3) \right] ds, \\ \delta_2 &= \frac{9k}{\pi R^3} \int_{\Gamma_R} \left[ (v_1 x_2 + v_2 x_1) - \frac{4}{3R^2}(v_1 x_2^3 + v_2 x_1^3) \right] ds.\end{aligned}$$

## 2 Transformations of the energy functional for a 2D Laplace equation

### 2.1 Derivation using Poisson kernel

Let us consider the bulk energy functional of the form

$$E(u) = \frac{1}{2} \int_{\Omega} \|\nabla u\|^2 dx, \quad (39)$$

where  $u$  satisfies inside the domain  $\Omega \subset \mathbb{R}^2$  the Laplace equation

$$\Delta u = 0 \quad (40)$$

with suitable boundary conditions. Our goal is to study the influence of the small circular hole of the variable radius  $\rho$  contained in the domain, for simplicity we put its center at  $\mathbf{x} = 0$ . we do not want to study variable domains, so we isolate this hole inside the ring  $C(\rho, R) = \{ \mathbf{x} \mid \rho < \|\mathbf{x}\| < R \}$  and replace  $E(u)$  by the equivalent expression over  $\Omega_R = \Omega \setminus \overline{B(R)}$ , with an additional boundary integral term over  $\Gamma_R = \partial B(R)$ . On  $\Gamma_\rho = \partial B(\rho)$  we assume homogeneous Neumann conditions.

In the first step we modify  $E(u)$ . Since

$$\int_{\Omega} \|\nabla u\|^2 dx = \int_{\Omega_R} \|\nabla u\|^2 dx + \int_{B(R)} \|\nabla u\|^2 dx = \int_{\Omega_R} \|\nabla u\|^2 dx + \int_{\Gamma_R} u \frac{\partial u}{\partial n} ds, \quad (41)$$

where  $\mathbf{n}$  is the outward normal vector on the boundary of  $B(R)$ , we may concentrate on the expression

$$E_R(u) = \int_{\Gamma_R} u \frac{\partial u}{\partial n} ds. \quad (42)$$



Here the values of  $\partial u / \partial n$  on  $\Gamma_R$  are given by the Dirichlet to Neumann mapping, namely the solution of

$$\Delta v = 0 \quad \text{in } B(R), \quad v = u \quad \text{on } \Gamma_R, \quad (43)$$

by means of the relation  $\partial u / \partial n = \partial v / \partial n$ .

The function  $v$  may be constructed with the help of Poisson kernel. If the pair  $(r, \phi)$  constitutes polar coordinates around 0, then

$$v(r, \phi) = \frac{1}{\pi} \int_{-\pi}^{\pi} u(R, \psi) \left[ \frac{1}{2} + \sum_{n=1}^{\infty} \left( \frac{r}{R} \right)^n \cos n(\phi - \psi) \right] d\psi \quad (44)$$

for any  $r \leq R$ . We shall assume for a moment that  $r < R$ . Differentiating (44) gives

$$\begin{aligned} \frac{\partial v}{\partial r}(r, \phi) &= \frac{1}{\pi} \int_{-\pi}^{\pi} u(R, \psi) \left[ \sum_{n=1}^{\infty} n \frac{1}{R} \left( \frac{r}{R} \right)^{n-1} \cos n(\phi - \psi) \right] d\psi \\ &= \frac{1}{\pi r} \int_{-\pi}^{\pi} u(R, \psi) \left[ \sum_{n=1}^{\infty} n \left( \frac{r}{R} \right)^n \cos n(\phi - \psi) \right] d\psi \\ &= -\frac{1}{\pi r} \int_{-\pi}^{\pi} u(R, \psi) \frac{\partial}{\partial \psi} \left[ \sum_{n=1}^{\infty} \left( \frac{r}{R} \right)^n \sin n(\phi - \psi) \right] d\psi \\ &= \frac{1}{\pi r} \int_{-\pi}^{\pi} \frac{\partial u}{\partial \psi}(R, \psi) \left[ \sum_{n=1}^{\infty} \left( \frac{r}{R} \right)^n \sin n(\phi - \psi) \right] d\psi. \end{aligned} \quad (45)$$

While integrating by parts we have used here the continuity of  $u$  over  $\Gamma_R$ . Applying the following trick (for  $|t| < 1$ ):

$$\begin{aligned} \sum_{n=1}^{\infty} t^n \sin n\alpha &= \frac{1}{2i} \sum_{n=1}^{\infty} [(te^{i\alpha})^n - (te^{-i\alpha})^n] \\ &= \frac{1}{2i} \left[ \frac{te^{i\alpha}}{1 - te^{i\alpha}} - \frac{te^{-i\alpha}}{1 - te^{-i\alpha}} \right] \\ &= \frac{t \sin \alpha}{1 - 2t \cos \alpha + t^2} = H(t, \alpha) \end{aligned}$$

we arrive at

$$\frac{\partial v}{\partial r}(r, \phi) = \frac{1}{\pi r} \int_{-\pi}^{\pi} \frac{\partial u}{\partial \psi}(R, \psi) H\left(\frac{r}{R}, \phi - \psi\right) d\psi. \quad (46)$$

We shall need also the value of  $\nabla v(0)$ . Let us rewrite (44) by means of Poisson kernel,

$$v(r, \phi) = \frac{1}{2\pi} \int_{-\pi}^{\pi} u(R, \psi) K\left(\frac{r}{R}, \phi - \psi\right) d\psi, \quad (47)$$

where

$$K(t, \alpha) = \frac{1 - t^2}{1 - 2t \cos \alpha + t^2}. \quad (48)$$

Now

$$\begin{aligned}\frac{\partial v}{\partial x_1}(0) &= \lim_{r \rightarrow 0+} \left[ \frac{\partial v}{\partial r}(r, \phi) \cos \phi - \frac{1}{r} \frac{\partial v}{\partial \phi}(r, \phi) \sin \phi \right], \\ \frac{\partial v}{\partial x_2}(0) &= \lim_{r \rightarrow 0+} \left[ \frac{\partial v}{\partial r}(r, \phi) \sin \phi + \frac{1}{r} \frac{\partial v}{\partial \phi}(r, \phi) \cos \phi \right],\end{aligned}\tag{49}$$

and

$$\begin{aligned}\lim_{r \rightarrow 0+} \frac{\partial K}{\partial r}(\frac{r}{R}, \phi - \psi) &= \frac{2}{R} \cos(\phi - \psi), \\ \lim_{r \rightarrow 0+} \frac{\partial K}{\partial r}(\frac{r}{R}, \phi - \psi) &= -\frac{2}{R} \sin(\phi - \psi).\end{aligned}$$

Hence, after differentiating (47) and substitution into (49), we obtain the final result

$$\begin{aligned}\frac{\partial u}{\partial x_1}(0) &= \frac{1}{\pi R} \int_{-\pi}^{\pi} u(R, \psi) \cos \psi \, d\psi = \frac{1}{\pi R^3} \int_{\Gamma_R} u x_1 \, ds, \\ \frac{\partial u}{\partial x_2}(0) &= \frac{1}{\pi R} \int_{-\pi}^{\pi} u(R, \psi) \sin \psi \, d\psi = \frac{1}{\pi R^3} \int_{\Gamma_R} u x_2 \, ds.\end{aligned}\tag{50}$$

Let us now return to  $E_R(u)$ . We consider  $E_r(u)$  given for  $r < R$  and  $\Gamma_r = \partial B(r)$  by

$$E_r(u) = \int_{\Gamma_r} u \frac{\partial u}{\partial r} \, ds = \int_{-\pi}^{\pi} u(r, \phi) \frac{\partial v}{\partial r}(r, \phi) r \, d\phi.\tag{51}$$

After substituting (46),

$$E_r(u) = \frac{1}{\pi} \int_{-\pi}^{\pi} \int_{-\pi}^{\pi} u(r, \phi) \frac{\partial u}{\partial \psi}(R, \phi) H(\frac{r}{R}, \phi - \psi) \, d\phi d\psi.\tag{52}$$

Now we observe that  $H$  may be expressed as

$$H(t, \alpha) = \frac{1}{2} \frac{\partial}{\partial \alpha} H_0(t, \alpha),\tag{53}$$

where

$$H_0(t, \alpha) = \log(1 - 2t \cos \alpha + t^2).$$

Therefore (52) transforms to

$$\begin{aligned}E_r(u) &= \frac{1}{2\pi} \int_{-\pi}^{\pi} \int_{-\pi}^{\pi} \frac{\partial u}{\partial \psi}(R, \phi) u(r, \phi) \frac{\partial}{\partial \phi} H_0(\frac{r}{R}, \phi - \psi) \, d\phi d\psi, \\ &= -\frac{1}{2\pi} \int_{-\pi}^{\pi} \int_{-\pi}^{\pi} \frac{\partial u}{\partial \psi}(R, \phi) \frac{\partial u}{\partial \phi}(r, \phi) H_0(\frac{r}{R}, \phi - \psi) \, d\phi d\psi.\end{aligned}\tag{54}$$

The function  $H_0(1, \alpha)$  has for  $\alpha = 0$  an integrable singularity of the type  $\log |\alpha|$ , so we may pass to the limit with  $r \rightarrow R$  obtaining

$$E_R(u) = -\frac{1}{2\pi} \int_{-\pi}^{\pi} \int_{-\pi}^{\pi} \frac{\partial u}{\partial \psi}(R, \psi) \frac{\partial u}{\partial \phi}(R, \phi) H_0(1, \phi - \psi) d\phi d\psi. \quad (55)$$

Now we want to take into account the influence of the small hole  $B(\rho)$ . Assuming the value of  $u$  on  $\Gamma_R$  given, it will change the value of  $\partial u / \partial n$ . In order to evaluate this change we shall use the asymptotic expansion derived in [35]. Namely, if  $v_\rho$  is the solution of

$$\Delta v_\rho = 0 \quad \text{in } B(R), \quad v_\rho = u \quad \text{on } \Gamma_R, \quad \frac{\partial v_\rho}{\partial n} = 0 \quad \text{on } \Gamma_\rho, \quad (56)$$

then, assuming  $\rho < \frac{1}{2}R$ , we have

$$\begin{aligned} v_\rho &= v + s_\rho + o(\rho^2), \\ \frac{\partial v_\rho}{\partial r} &= \frac{\partial v}{\partial r} + \frac{\partial s_\rho}{\partial r} + o(\rho^2), \end{aligned} \quad (57)$$

for  $r > \frac{1}{2}R$ . Here

$$s_\rho = \frac{\rho^2}{r} \left( \frac{\partial v}{\partial x_1}(0) \cos \phi + \frac{\partial v}{\partial x_2}(0) \sin \phi \right), \quad (58)$$

and therefore

$$\frac{\partial s_\rho}{\partial r} = -\frac{\rho^2}{r^2} \left( \frac{\partial v}{\partial x_1}(0) \cos \phi + \frac{\partial v}{\partial x_2}(0) \sin \phi \right). \quad (59)$$

Substituting this correction into (51) gives

$$\begin{aligned} \delta E_r(u) &= -\frac{\rho^2}{r^2} \int_{-\pi}^{\pi} u(r, \phi) \left[ \frac{\partial v}{\partial x_1}(0) \cos \phi + \frac{\partial v}{\partial x_2}(0) \sin \phi \right] r d\phi \\ &= -\frac{\rho^2}{r} \left[ \frac{\partial v}{\partial x_1}(0) \int_{-\pi}^{\pi} u(r, \phi) \cos \phi d\phi + \frac{\partial v}{\partial x_2}(0) \int_{-\pi}^{\pi} u(r, \phi) \sin \phi d\phi \right]. \end{aligned}$$

Now passing to the limit with  $r \rightarrow R$  and taking into account that  $u = v$ , we obtain in view of (50)

$$\delta E_R(u) = -\frac{\rho^2}{\pi R^6} \left[ \left( \int_{\Gamma_R} u x_1 ds \right)^2 + \left( \int_{\Gamma_R} u x_2 ds \right)^2 \right]. \quad (60)$$

This allows us to consider the following approximation. Let  $u_\rho$  satisfies (40) inside  $\Omega_\rho = \Omega \setminus \overline{B(\rho)}$  and take

$$E(\rho; u_\rho) = \frac{1}{2} \int_{\Omega_\rho} \|\nabla u_\rho\|^2 dx. \quad (61)$$

Then we may consider  $u_\rho$  in a fixed domain  $\Omega_R$  and add boundary terms

$$E(\rho; u_\rho) = \frac{1}{2} \int_{\Omega_R} \|\nabla u_\rho\|^2 dx + \frac{1}{2} E_R(u_\rho) + \frac{1}{2} \delta E_R(u_\rho) + o(\rho^2). \quad (62)$$

This means that in fact we have to do with some function  $w_R$ , satisfying (40) in  $\Omega_R$ , for which the bulk energy is given by

$$\begin{aligned} E(\rho; w_R) = & \frac{1}{2} \int_{\Omega_R} \|\nabla w_R\|^2 dx \\ & - \frac{1}{4\pi} \int_{-\pi}^{\pi} \int_{-\pi}^{\pi} \frac{\partial w_R}{\partial \psi}(R, \psi) \frac{\partial w_R}{\partial \phi}(R, \phi) H_0(1, \phi - \psi) d\phi d\psi \\ & - \frac{\rho^2}{2\pi R^6} \left[ \left( \int_{\Gamma_R} w_R x_1 ds \right)^2 + \left( \int_{\Gamma_R} w_R x_2 ds \right)^2 \right]. \end{aligned} \quad (63)$$

The first two terms give exact expression for the intact domain  $\Omega$ , and the third approximates the influence of the small hole. All of them are quadratic forms of  $w_R$ .

The representation (63) allows us to transform the task from solving the boundary value problem for Laplace equation in variable domain  $\Omega_\rho$  to the same problem in the fixed domain  $\Omega_R$ , but with energy parametrized by  $\rho$ . In fact, since the first terms represent energy for the whole of  $\Omega$ , we may even avoid using  $\Omega_R$  (and the remeshing of  $\Omega$  in order to get discretization of  $\Omega_R$ ). This results from observation that outside  $B(R)$  the function  $w_R$  coincides with  $w$  given by the solution of the boundary value problem with bulk energy given by

$$E_0(\rho; w) = \frac{1}{2} \int_{\Omega} \|\nabla w\|^2 dx - \frac{\rho^2}{2\pi R^6} \left[ \left( \int_{\Gamma_R} w x_1 ds \right)^2 + \left( \int_{\Gamma_R} w x_2 ds \right)^2 \right]. \quad (64)$$

The weak form of this problem is easily obtained by taking variation of  $E_0(\rho; w)$ , adding external work and using boundary conditions on  $\partial\Omega$ . As we see, modification of bulk energy accounts to introducing source term concentrated on the fixed circle  $\Gamma_R$  and parametrized by  $\rho$ .

The last formula may be expressed also in another form. Let us denote by  $e_u(\mathbf{x})$  the energy density at the point  $\mathbf{x}$ ,

$$e_u(\mathbf{x}_0) = \|\nabla u(\mathbf{x}_0)\|^2.$$

If the function  $u$  is harmonic in  $B(R)$ , then the expressions for derivatives

$$u_{/1}(\mathbf{x}_0) = \frac{1}{\pi R^3} \int_{\Gamma_R(\mathbf{x}_0)} u \cdot (x_1 - x_{1,0}) ds, \quad u_{/2}(\mathbf{x}_0) = \frac{1}{\pi R^3} \int_{\Gamma_R(\mathbf{x}_0)} u \cdot (x_2 - x_{2,0}) ds.$$

are exact. In view of this, formula (64) can be rewritten in the equivalent form

$$E_0(\rho; w) = \frac{1}{2} \int_{\Omega} \|\nabla w\|^2 dx - \frac{1}{2} \pi \rho^2 e_w(0). \quad (65)$$

## 2.2 Numerical tests of the transformed energy approach

We consider the test problem

$$\begin{aligned} \Delta u_\rho &= 0 \quad \text{in } \Omega_\rho = [0, 1] \times [0, 1] \setminus \overline{B(\mathbf{x}_0, \rho)}, \\ u_\rho &= \begin{cases} 0 & \text{for } x_1 = 0, \\ 1 & \text{for } x_1 = 1, \\ x_1^2 & \text{for } x_2 = 0, \\ \sqrt{x_1} & \text{for } x_2 = 1. \end{cases} \\ \frac{\partial u_\rho}{\partial n} &= 0 \quad \text{on } \partial B(\mathbf{x}_0, \rho). \end{aligned} \tag{66}$$

Three types of approximations are used. Here by  $u$  we denote the solution in the domain without void.

1. Double correction: the function  $u_\rho$  is represented in the form

$$u_\rho = u + s_\rho(u) + p_\rho + s_\rho(p_\rho),$$

where  $s_\rho(u)$  is the first correction for  $u$ . However,  $s_\rho(u)$  disturbs the boundary conditions on  $\partial\Omega$  and we introduce second corrections. The function  $p_\rho$  solves the Laplace equation with the boundary condition  $p_\rho = -s_\rho(u)$  on  $\partial\Omega$ , and then again  $s_\rho(p_\rho)$  nullifies the Neumann condition on the boundary of the void. We consider this solution as nearly exact in  $\Omega_\rho$ .

2. "Exact" solution: here the right-hand side is augmented by the expression containing derivatives of Dirac's delta,

$$\Delta u_\rho = 2\pi\rho^2(\nabla\delta(\mathbf{x} - \mathbf{x}_0) \cdot \nabla u).$$

In theory this solution is also exact in  $\Omega_\rho$  up to the higher than 2 powers of  $\rho$ , see [24], but of course there are difficulties with numerical approximation.

3. Solution obtained by the modification of the energy term given by (65). This should be exact up to the higher than 2 powers of  $\rho$  outside the ring of the radius  $R > \rho$ .

The hole was positioned at  $\mathbf{x}_0 = [0.5, 0.7]$  and had the radius  $\rho = 0.05$ . The figures show the results of computations for different positions of the void and ratios  $R/\rho$ , in the form of sections through the surface  $u(x_1) = u(x_1, x_2^0)$  along the line  $x_2 = x_2^0$ , i.e. going through the middle of the hole. Other sections look very similar. We may conclude, that the modified energy allows, in concordance with the theory developed in last subsection, to compute accurately the solution outside the ring  $\Gamma_R$ .

Moreover we have seen a very good agreement of the solution with the singular right-hand side with the nearly exact solution produced by double correction. This justifies the use of singular solutions as an approximation of real ones, what will be exploited in the next sections.

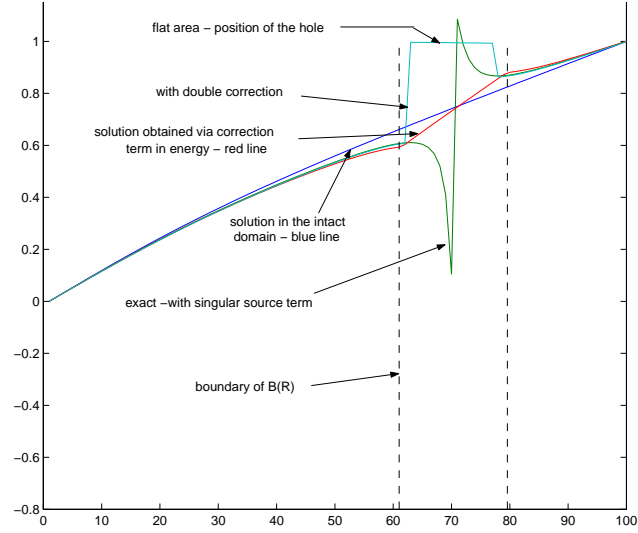


Figure 1: Comparison of solutions for  $R = 1.2\rho$ . Visible slight loss of accuracy near boundary of the ring, due to the small ratio of the radii.

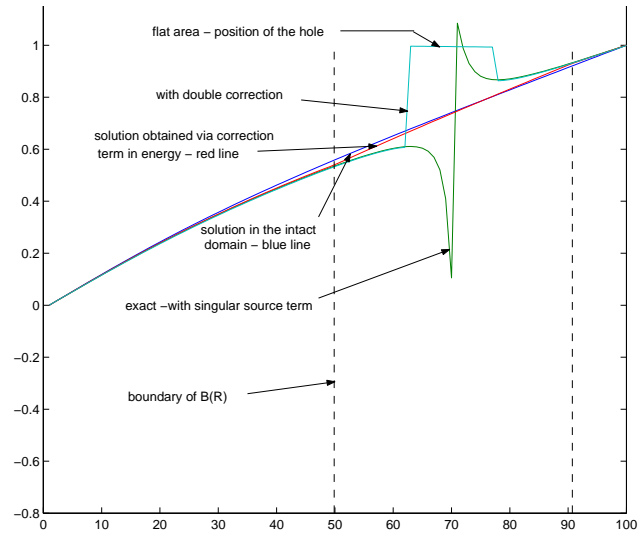


Figure 2: The same as in Fig. 1, but with  $R = 2.5\rho$ .

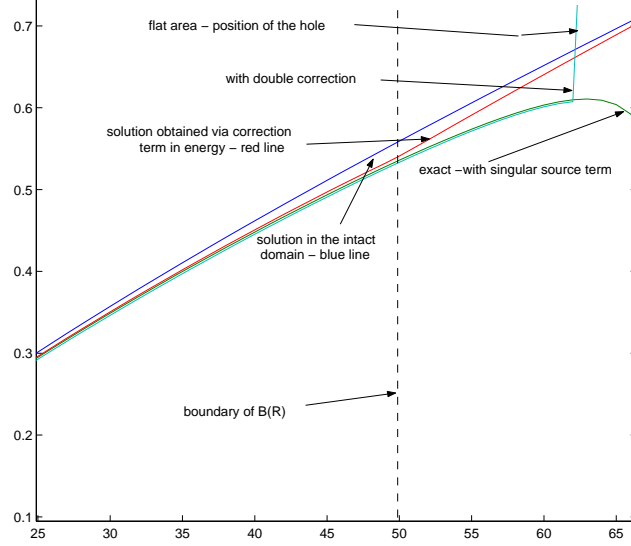


Figure 3: The blow-up of the important fragment of Fig. 2.

### 3 Transformations of the energy functional for the 2D elasticity system

#### 3.1 Using Poisson kernel for computing strain

As it turns out, similar reasoning may be carried out in case of the 2D elasticity system, even if it is much more complicated. In absence of volume forces such a system has a form

$$\begin{aligned}\mu\Delta u_1 + (\lambda + \mu)(u_{1/1,1} + u_{2/1,2}) &= 0, \\ \mu\Delta u_2 + (\lambda + \mu)(u_{1/1,2} + u_{2/2,2}) &= 0,\end{aligned}\tag{67}$$

where  $\mathbf{u} = (u_1, u_2)^T$  denotes the displacement and  $\lambda, \mu$  are Lamé constants. We shall use also the usual notation for the symmetric strain tensor  $\epsilon = [\epsilon_{ij}]$ ,  $\epsilon_{11} = u_{1/1}$ ,  $\epsilon_{22} = u_{2/2}$ ,  $\gamma_{12} = 2\epsilon_{12} = u_{1/2} + u_{2/1}$  as well as stress tensor  $\sigma = [\sigma_{ij}]$ . The Hooke's law

$$\sigma_{11} = (\lambda + 2\mu)\epsilon_{11} + \lambda\epsilon_{22}, \quad \sigma_{22} = \lambda\epsilon_{11} + (\lambda + 2\mu)\epsilon_{22}, \quad \sigma_{12} = \mu\gamma_{12} = 2\mu\epsilon_{12}$$

links both objects. In these terms (67) reduces to

$$\nabla \cdot \sigma(\mathbf{u}) = 0.\tag{68}$$

For such a system there exists an analog to the Poisson kernel, see [6]. It is a matrix  $\mathbf{G}(\mathbf{x}, \mathbf{y})$  allowing us to express the values of the solution inside the circle  $\Gamma_R(\mathbf{x}_0)$  by means of its

values on the circumference:

$$\mathbf{u}(\mathbf{x}) = -\frac{1}{\pi} \int_{\Gamma_R(\mathbf{x}_0)} \mathbf{G}(\mathbf{x} - \mathbf{x}_0, \mathbf{y} - \mathbf{x}_0) \cdot \mathbf{u}(\mathbf{y}) ds_y. \quad (69)$$

Let us denote  $\mathbf{I}$  – identity matrix and

$$k = \frac{\lambda + \mu}{\lambda + 3\mu}.$$

Then  $\mathbf{G}(\mathbf{x}, \mathbf{y})$  has a form

$$\mathbf{G}(\mathbf{x}, \mathbf{y}) = \mathbf{\Gamma}(\mathbf{x}, \mathbf{y}) + \mathbf{A}(\mathbf{x}, \mathbf{y}), \quad (70)$$

where

$$\mathbf{\Gamma}(\mathbf{x}, \mathbf{y}) = \left( (1-k)\mathbf{I} + 2k \begin{bmatrix} \left(\frac{\partial d}{\partial x_1}\right)^2 & \frac{\partial d}{\partial x_1} \frac{\partial d}{\partial x_2} \\ \frac{\partial d}{\partial x_1} \frac{\partial d}{\partial x_2} & \left(\frac{\partial d}{\partial x_2}\right)^2 \end{bmatrix} \right) \frac{\partial}{\partial \mathbf{n}_y} \log \frac{1}{d}, \quad (71)$$

$$\mathbf{A}(\mathbf{x}, \mathbf{y}) = \frac{1}{2R} \left( (1-k)\mathbf{I} - k \begin{bmatrix} \frac{x_1 y_1 - x_2 y_2}{R^2} - 1 & \frac{x_1 y_2 + x_2 y_1}{R^2} \\ \frac{x_1 y_2 + x_2 y_1}{R^2} & -1 - \frac{x_1 y_1 - x_2 y_2}{R^2} \end{bmatrix} \right), \quad (72)$$

and  $d = d(\mathbf{x}, \mathbf{y}) = \|\mathbf{x} - \mathbf{y}\|$ .

*From now on we shall assume that  $\mathbf{x}_0 = 0$ . This greatly simplifies formulae without loss of generality.*

Using the representation of displacement as given by (69) we may compute the values of its derivatives at 0. Before writing down the result, we must introduce some notation. Let us define  $I_1(k, l)$  and  $I_2(k, l)$  as

$$I_1(k, l) = \frac{1}{\alpha(k, l)} \int_{\Gamma_R} u_1 x_1^k x_2^l ds, \quad I_2(k, l) = \frac{1}{\beta(k, l)} \int_{\Gamma_R} u_2 x_1^k x_2^l ds, \quad (73)$$

where

$$\begin{aligned} \alpha(k, l) &= R^{k+l+2} \int_0^{2\pi} \cos^{k+1} \phi \sin^l \phi d\phi, \\ \beta(k, l) &= R^{k+l+2} \int_0^{2\pi} \cos^k \phi \sin^{l+1} \phi d\phi, \end{aligned}$$

whenever these expressions make sense, i.e. if  $k$  is odd and  $l$  even or vice versa. Observe that  $\alpha(k, 0) = \beta(0, k)$  and

$$\alpha(1, 0) = \pi R^3, \quad \alpha(3, 0) = \frac{3}{4}\pi R^5, \quad \alpha(1, 2) = \frac{1}{4}\pi R^5, \quad \alpha(5, 0) = \frac{5}{8}\pi R^7, \quad \alpha(3, 2) = \frac{1}{8}\pi R^7$$



and so on. Furthermore, let

$$\begin{aligned}\delta_1 &= 9k ([I_1(1, 0) - I_2(0, 1)] - [I_1(3, 0) - I_2(0, 3)]), \\ \delta_2 &= 9k ([I_1(0, 1) + I_2(1, 0)] - [I_1(0, 3) + I_2(3, 0)]).\end{aligned}\quad (74)$$

In terms of these symbols one may obtain, after very lengthy calculations, the formulae for the values of strain components at the point  $\mathbf{x}_0 = 0$  which will constitute the basis of our energy transformations:

$$\begin{aligned}\epsilon_{11} + \epsilon_{22} &= I_1(1, 0) + I_2(0, 1), \\ \epsilon_{11} - \epsilon_{22} &= I_1(1, 0) - I_2(0, 1) - \delta_1, \\ \gamma_{12} &= I_1(0, 1) + I_2(1, 0) + \delta_2.\end{aligned}\quad (75)$$

Let us recall also the expression for the elastic energy density at the same point,

$$e_{\mathbf{u}}(0) = \frac{1}{2} \boldsymbol{\sigma} : \boldsymbol{\epsilon} = \frac{1}{2} [(\lambda + \mu)(\epsilon_{11} + \epsilon_{22})^2 + \mu(\epsilon_{11} - \epsilon_{22})^2 + \mu\gamma_{12}^2]. \quad (76)$$

### 3.2 Rationale behind the notation and another derivation

The origin of symbols  $I_1(k, l)$ ,  $I_2(k, l)$  and their role in calculation of strains may be explained also in another way, leading to a general procedure. To this means we shall use the formal series method. We assume displacement around  $\mathbf{x}_0 = 0$  in the form of the power series

$$\begin{aligned}u_1 &= a_{0,0} + a_{1,0}x_1 + a_{0,1}x_2 + \dots = \sum_{k=0}^{\infty} \sum_{l=0}^k a_{k-l,l} x_1^{k-l} x_2^l, \\ u_2 &= b_{0,0} + b_{1,0}x_1 + b_{0,1}x_2 + \dots = \sum_{k=0}^{\infty} \sum_{l=0}^k b_{k-l,l} x_1^{k-l} x_2^l.\end{aligned}\quad (77)$$

We are interested in a possibly accurate computation of

$$\begin{aligned}\epsilon_{11} &= u_{1/1}(0) = a_{1,0}, \\ \epsilon_{22} &= u_{2/2}(0) = b_{0,1}, \\ \gamma_{12} &= u_{1/2}(0) + u_{2/1}(0) = a_{0,1} + b_{1,0}.\end{aligned}\quad (78)$$

Since the equations (67) must be satisfied at  $\mathbf{x} = 0$ , after differentiating the series and substitution we have

$$\begin{aligned}2\mu(a_{2,0} + a_{0,2}) + (\lambda + \mu)(2a_{2,0} + b_{1,1}) &= 0, \\ 2\mu(b_{2,0} + b_{0,2}) + (\lambda + \mu)(a_{1,1} + 2b_{0,2}) &= 0.\end{aligned}\quad (79)$$

Differentiating the first equation in (67) with respect to  $x_1$  and the second with respect to  $x_2$  we obtain in a similar way

$$\begin{aligned}\mu(6a_{3,0} + 2a_{1,2}) + (\lambda + \mu)(6a_{3,0} + 2b_{2,1}) &= 0, \\ \mu(2b_{2,1} + 6b_{0,3}) + (\lambda + \mu)(2a_{1,2} + 6b_{0,3}) &= 0.\end{aligned}\quad (80)$$

By summig the above we get

$$3a_{3,0} + a_{1,2} + b_{2,1} + 3b_{0,3} = 0. \quad (81)$$

Similar relations may be obtained for higher coefficients of both series, by more differentiations.

In the first step we shall compute  $\epsilon_{11} + \epsilon_{22}$ . We integrate (77) over  $\Gamma_R$  getting

$$\begin{aligned} \int_{\Gamma_R} u_1 x_1 ds &= \alpha(1,0)a_{1,0} + \alpha(3,0)a_{3,0} + \alpha(1,2)a_{1,2} + \dots \\ \int_{\Gamma_R} u_2 x_2 ds &= \beta(0,1)b_{0,1} + \beta(0,3)b_{0,3} + \beta(2,1)b_{2,1} + \dots \end{aligned} \quad (82)$$

Adding these relations and using properties of functions  $\alpha, \beta$  gives

$$I_1(1,0) + I_2(0,1) = a_{1,0} + b_{0,1} + \frac{R^2}{4}(3a_{3,0} + a_{1,2} + b_{2,1} + 3b_{0,3}) + \dots \quad (83)$$

The term in parenthesis vanishes due to (81) and in a similar way one can check that farther terms disappear as well. This gives *exact* relation

$$\epsilon_{11} + \epsilon_{22} = a_{1,0} + b_{0,1} = I_1(1,0) + I_2(0,1) \quad (84)$$

as expected. On the way we have seen the origin of  $I_1(1,0)$  and  $I_2(0,1)$ , which appear naturally due to integration of the series.

In the next step we compute the difference

$$\epsilon_{11} - \epsilon_{22} = a_{1,0} - b_{0,1}.$$

Proceeding similarly as above we get

$$I_1(1,0) - I_2(0,1) = a_{1,0} - b_{0,1} + \frac{R^2}{4}(3a_{3,0} + a_{1,2} - b_{2,1} - 3b_{0,3}) + \dots \quad (85)$$

and by substracting the equations in (80)

$$\mu(3a_{3,0} + a_{1,2} - b_{2,1} - 3b_{0,3}) + (\lambda + \mu)(3a_{3,0} - a_{1,2} + b_{2,1} - 3b_{0,3}) = 0. \quad (86)$$

This is insufficient to cancel farther terms in (85) and we need additional equations. To this end we compute higher order integrals

$$\begin{aligned} \int_{\Gamma_R} u_1 x_1^3 ds &= \alpha(3,0)a_{1,0} + \alpha(5,0)a_{3,0} + \alpha(3,2)a_{1,2} + \dots \\ \int_{\Gamma_R} u_2 x_2^3 ds &= \beta(0,3)b_{0,1} + \beta(0,5)b_{0,3} + \beta(2,3)b_{2,1} + \dots \end{aligned} \quad (87)$$

From the above follows

$$I_1(3, 0) - I_2(0, 3) = a_{1,0} - b_{0,1} + \frac{R^2}{6}(5a_{3,0} + a_{1,2} - b_{2,1} - 5b_{0,3}) + \dots \quad (88)$$

where again we see the origins of definitions for  $I_1(3, 0)$  and  $I_2(0, 3)$ .

Combining (88) with (85) as well as (86) gives, after some calculations, the second formula in (75). In the same way one can get the third formula for  $\gamma_{12} = a_{0,1} + b_{1,0}$ .

Observe, that if displacement components were harmonic, then the corrections  $\delta_1$  and  $\delta_2$  would vanish, as we have seen in section concerning the Laplace equation. In practice, as we shall see from numerical experiments, they are very small, but not negligible.

We have presented this alternative derivation because it may have some advantages. The knowledge of Poisson kernel is not required, and anisotropic nonhomogeneous equations fall into the same schema, assuming sufficient differentiability of volume forces. In such a case on the right-hand side of equations there appear series representing these forces.

### 3.3 Distortion of the stress field caused by small circular hole

We shall recall here some formulae describing the stress field around circular hole in the infinite 2-D elastic medium. If we assume that at infinity only  $\sigma_{11}$  is not zero, and the hole  $B(\rho)$  is centered around origin, then the stresses for  $r \geq \rho$  have the form

$$\begin{aligned} \sigma_{rr} &= \frac{1}{2}\sigma_{11} \left[ \left(1 - \frac{\rho^2}{r^2}\right) + \left(1 - 4\frac{\rho^2}{r^2} + 3\frac{\rho^4}{r^4}\right) \cos 2\phi \right], \\ \sigma_{\phi\phi} &= \frac{1}{2}\sigma_{11} \left[ \left(1 + \frac{\rho^2}{r^2}\right) - \left(1 + 3\frac{\rho^4}{r^4}\right) \cos 2\phi \right], \\ \sigma_{r\phi} &= -\frac{1}{2}\sigma_{11} \left(1 + 2\frac{\rho^2}{r^2} - 3\frac{\rho^4}{r^4}\right) \sin 2\phi. \end{aligned} \quad (89)$$

Here  $(r, \phi)$  constitute the polar coordinate system around origin and the  $\sigma$ -components are give in the orthogonal coordinates defined by  $\{e_r, e_\phi\}$ , with base versors at any given point directed along radius and perpendicularly to it, anticlockwise.

Using these expressions we may immediately construct the solution corresponding to nonzero  $\sigma_{22}$  at infinity, by substituting  $\phi := \phi + \frac{\pi}{2}$ ,  $\sigma_{11} := \sigma_{22}$  (exchange of axis):

$$\begin{aligned} \sigma_{rr} &= \frac{1}{2}\sigma_{22} \left[ \left(1 - \frac{\rho^2}{r^2}\right) - \left(1 - 4\frac{\rho^2}{r^2} + 3\frac{\rho^4}{r^4}\right) \cos 2\phi \right], \\ \sigma_{\phi\phi} &= \frac{1}{2}\sigma_{22} \left[ \left(1 + \frac{\rho^2}{r^2}\right) + \left(1 + 3\frac{\rho^4}{r^4}\right) \cos 2\phi \right], \\ \sigma_{r\phi} &= \frac{1}{2}\sigma_{22} \left(1 + 2\frac{\rho^2}{r^2} - 3\frac{\rho^4}{r^4}\right) \sin 2\phi. \end{aligned} \quad (90)$$

Furthermore, we may exploit the fact that the pure shear stress  $\sigma_{12}$  is equivalent to simultaneous stretching and compression with the same intensity  $\sigma_{12}$  and  $-\sigma_{12}$ , but along

the axis rotated by the angle  $\pi/4$ . Thus we make substitutions  $\phi := \phi + \frac{\pi}{4}$ ,  $\sigma_{11} := \sigma_{12}$ , then  $\phi := \phi - \frac{\pi}{4}$ ,  $\sigma_{11} := -\sigma_{12}$  in (89) and add both solutions together obtaining:

$$\begin{aligned}\sigma_{rr} &= \sigma_{12} \left(1 - 4\frac{\rho^2}{r^2} + 3\frac{\rho^4}{r^4}\right) \sin 2\phi, \\ \sigma_{\phi\phi} &= \sigma_{12} \left(1 + 3\frac{\rho^4}{r^4}\right) \sin 2\phi, \\ \sigma_{r\phi} &= \sigma_{12} \left(1 + 2\frac{\rho^2}{r^2} - 3\frac{\rho^4}{r^4}\right) \cos 2\phi.\end{aligned}\tag{91}$$

Let us now denote

$$\sigma_I = \frac{1}{2}(\sigma_{11} + \sigma_{22}), \quad \sigma_{II} = \frac{1}{2}(\sigma_{11} - \sigma_{22}).\tag{92}$$

Then adding (89),(90),(91) gives the solution corresponding to the general stress field at infinity:

$$\begin{aligned}\sigma_{rr} &= \sigma_I + \sigma_{II} \cos 2\phi + \sigma_{12} \sin 2\phi \\ &\quad - \sigma_I \frac{\rho^2}{r^2} - \sigma_{II} \left(4\frac{\rho^2}{r^2} - 3\frac{\rho^4}{r^4}\right) \cos 2\phi - \sigma_{12} \left(4\frac{\rho^2}{r^2} - 3\frac{\rho^4}{r^4}\right) \sin 2\phi, \\ \sigma_{\phi\phi} &= \sigma_I - \sigma_{II} \cos 2\phi - \sigma_{12} \sin 2\phi \\ &\quad + \sigma_I \frac{\rho^2}{r^2} - 3\sigma_{II} \frac{\rho^4}{r^4} \cos 2\phi - 3\sigma_{12} \frac{\rho^4}{r^4} \sin 2\phi, \\ \sigma_{r\phi} &= -\sigma_{II} \sin 2\phi + \sigma_{12} \cos 2\phi \\ &\quad - \sigma_{II} \left(2\frac{\rho^2}{r^2} - 3\frac{\rho^4}{r^4}\right) \sin 2\phi + \sigma_{12} \left(2\frac{\rho^2}{r^2} - 3\frac{\rho^4}{r^4}\right) \cos 2\phi.\end{aligned}\tag{93}$$

Recalling the rules for the transformation of stresses under rotation of the coordinate system, we get the distortion of the stress due to the circular hole:

$$\begin{aligned}\hat{\sigma}_{rr} &= -\sigma_I \frac{\rho^2}{r^2} - \sigma_{II} \left(4\frac{\rho^2}{r^2} - 3\frac{\rho^4}{r^4}\right) \cos 2\phi - \sigma_{12} \left(4\frac{\rho^2}{r^2} - 3\frac{\rho^4}{r^4}\right) \sin 2\phi, \\ \hat{\sigma}_{\phi\phi} &= \sigma_I \frac{\rho^2}{r^2} - 3\sigma_{II} \frac{\rho^4}{r^4} \cos 2\phi - 3\sigma_{12} \frac{\rho^4}{r^4} \sin 2\phi, \\ \hat{\sigma}_{r\phi} &= -\sigma_{II} \left(2\frac{\rho^2}{r^2} - 3\frac{\rho^4}{r^4}\right) \sin 2\phi + \sigma_{12} \left(2\frac{\rho^2}{r^2} - 3\frac{\rho^4}{r^4}\right) \cos 2\phi.\end{aligned}\tag{94}$$

### 3.4 Transformation of the energy functional

Now we shall consider the contribution, in the absence of volume forces, of the energy integral over the circle surrounding the origin (i.e. the potential location of the small hole)

$$e_R(\mathbf{u}) = \frac{1}{2} \int_{B(R)} (\boldsymbol{\sigma} : \boldsymbol{\epsilon}) dx = \frac{1}{2} \int_{\Gamma_R} \mathbf{u}^T (\boldsymbol{\sigma} \cdot \mathbf{n}) ds\tag{95}$$

to the global elastic energy. Similarly as in the case of Laplace equation, we shall leave the displacement as is and consider the distortion to the stress field caused by introducing the small hole. Due to (94) it may be expressed as

$$\delta e_R = \frac{1}{2} \int_{\Gamma_R} \mathbf{u}^T (\hat{\boldsymbol{\sigma}} \cdot \mathbf{n}) ds. \quad (96)$$

At every point on the  $\Gamma_R$  we shall use the same coordinate system  $\{\mathbf{e}_r, \mathbf{e}_\phi\}$  as in the last section. In this system  $\mathbf{u} = [u_r, u_\phi]^T$ ,  $\mathbf{n} = [1, 0]^T$ . As a result, we have to compute the integral

$$\delta e_R = \frac{1}{2} \int_{\Gamma_R} (\hat{\sigma}_{rr} u_r + \hat{\sigma}_{r\phi} u_\phi) ds. \quad (97)$$

Now we observe that  $x_1^2 + x_2^2 = R^2$  on  $\Gamma_R$  and

$$\begin{aligned} u_r &= \frac{1}{R}(u_1 x_1 + u_2 x_2) & u_\phi &= \frac{1}{R}(-u_1 x_2 + u_2 x_1), \\ \sin \phi &= \frac{1}{R} x_2 & \cos \phi &= \frac{1}{R} x_1. \end{aligned}$$

To simplify the calculations we introduce notations:

$$\begin{aligned} f &= I(1, 0) + I(0, 1), & a &= I(1, 0) - I(0, 1), & b &= I(3, 0) - I(0, 3), \\ c &= I(0, 1) + I(1, 0), & d &= I(0, 3) - I(3, 0). \end{aligned}$$

In these terms

$$\begin{aligned} \int_{\Gamma_R} u_r ds &= \pi R^2 f, \\ \int_{\Gamma_R} u_r \cos 2\phi ds &= \pi R^2 \left(\frac{3}{2}b - a\right), \\ \int_{\Gamma_R} u_r \sin 2\phi ds &= \pi R^2 \left(2c - \frac{3}{2}d\right), \\ \int_{\Gamma_R} u_\phi \cos 2\phi ds &= \pi R^2 \left(\frac{3}{2}b - 2a\right), \\ \int_{\Gamma_R} u_\phi \sin 2\phi ds &= \pi R^2 \left(\frac{3}{2}d - c\right). \end{aligned} \quad (98)$$

Now, due to (74),(75),

$$\begin{aligned} f &= \epsilon_{11} + \epsilon_{22}, & a &= \epsilon_{11} - \epsilon_{22} + \delta_1, & b &= \epsilon_{11} - \epsilon_{22} + \left(1 - \frac{1}{9k}\right)\delta_1, \\ c &= \gamma_{12} - \delta_2, & d &= \gamma_{12} - \left(1 + \frac{1}{9k}\right)\delta_2. \end{aligned}$$

Substituting this into (97) gives

$$\delta e_R = -\frac{1}{2}\pi\rho^2 \left[ \sigma_I(\epsilon_{11} + \epsilon_{22}) + \sigma_{II}(\epsilon_{11} - \epsilon_{22}) + \sigma_{12}\gamma_{12} + \left(1 - \frac{1}{k} + \frac{\rho^2}{R^2} \frac{1}{k}\right) (\sigma_{II}\delta_1 - \sigma_{12}\delta_2) \right]. \quad (99)$$

Since from Hooke's law follows

$$\sigma_I = (\lambda + \mu)(\epsilon_{11} + \epsilon_{22}), \quad \sigma_{II} = \mu(\epsilon_{11} - \epsilon_{22}), \quad \sigma_{12} = \mu\gamma_{12}$$

then, because of (76),

$$\delta e_R = -\pi\rho^2 e_u(0) - \frac{1}{2}\pi\rho^2 \left[ \left(1 - \frac{1}{k} + \frac{\rho^2}{R^2} \frac{1}{k}\right) (\sigma_{II}\delta_1 - \sigma_{12}\delta_2) \right]. \quad (100)$$

This makes it different from the Laplace equation case, where the additional term vanished, see (65). Observe, that in order to solve the elasticity problem in the domain containing the hole with accuracy (outside  $\Gamma_R$ ) up to  $o(\rho^2)$ , we don't need, due to (100), the solution in the intact domain. Simultaneously all the terms in (100) are quadratic with respect to  $\mathbf{u}$  and introduce no difficulty into numerical procedures.

If we restrict ourself to the terms depending strictly on  $\rho^2$  and take into account the value of  $k$ , the energy corrections take on the form

$$\delta e_R = -\pi\rho^2 e_u(0) - \pi\rho^2 \frac{\mu}{\lambda + 3\mu} (\sigma_{II}\delta_1 - \sigma_{12}\delta_2). \quad (101)$$

In order to make clear that the energy correction is indeed an integral bilinear form of  $\mathbf{u}$  defined over  $\Gamma_R$ , we collect below the dependences given by (74),(75),(76) and write down the explicit expression for the terms appearing in (101):

$$\begin{aligned} \epsilon_{11} + \epsilon_{22} &= \frac{1}{\pi R^3} \int_{\Gamma_R} (u_1 x_1 + u_2 x_2) ds, \\ \epsilon_{11} - \epsilon_{22} &= \frac{1}{\pi R^3} \int_{\Gamma_R} \left[ (1 - 9k)(u_1 x_1 - u_2 x_2) + \frac{12k}{R^2} (u_1 x_1^3 - u_2 x_2^3) \right] ds, \\ \gamma_{12} &= \frac{1}{\pi R^3} \int_{\Gamma_R} \left[ (1 + 9k)(u_1 x_2 + u_2 x_1) - \frac{12k}{R^2} (u_1 x_2^3 + u_2 x_1^3) \right] ds, \\ \delta_1 &= \frac{9k}{\pi R^3} \int_{\Gamma_R} \left[ (u_1 x_1 - u_2 x_2) - \frac{4}{3R^2} (u_1 x_1^3 - u_2 x_2^3) \right] ds, \\ \delta_2 &= \frac{9k}{\pi R^3} \int_{\Gamma_R} \left[ (u_1 x_2 + u_2 x_1) - \frac{4}{3R^2} (u_1 x_2^3 + u_2 x_1^3) \right] ds. \end{aligned}$$

These expressions are easy to compute numerically, but unfortunately the correction formula is not so compact as in the Laplace operator case.

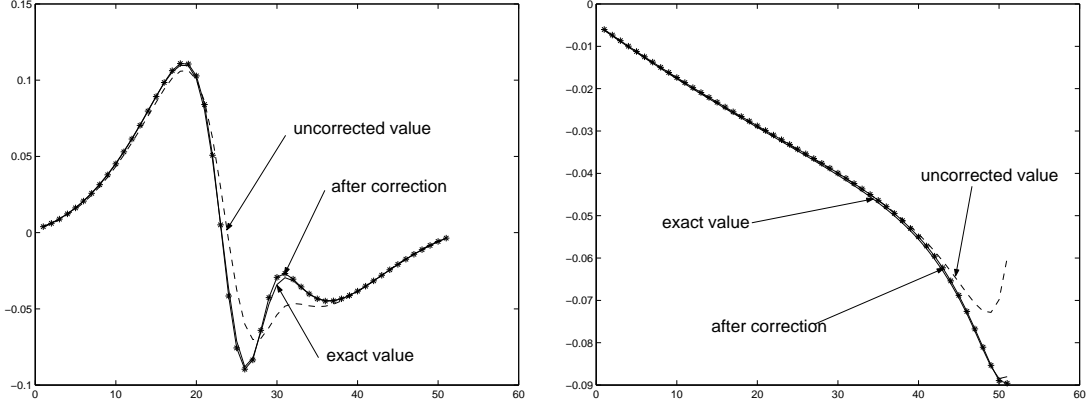


Figure 4: Graphs of the  $\epsilon_{11}$  stress component along the horizontal line near the upper edge (left) and the vertical line in the middle.

### 3.5 Testing the accuracy of the corrections

In this section we shall see how the formulae for stress components behave in practice. As a test example we take the square domain  $\Omega = [0, 1] \times [0, 1]$  discretized with  $100 \times 100$  grid using bilinear finite elements. It is fixed on  $\Gamma_0 = [0, 1] \times \{0\}$  and loaded with the traction  $T = [15, 15]$  (all after rescaling) along part of the upper edge  $\Gamma_1 = [0.45, 0.55] \times \{1\}$ . The Lamé coefficients are equal,  $\lambda = \mu$ . This defines the problem up to the proportionality factor. The horizontal line mentioned in the captions links the points  $[0.075, 0.925]$  and  $[0.925, 0.925]$ , while the vertical one connects  $[0.5, 0.075]$  and  $[0.5, 0.925]$ . They have 50 nodes, in which we compute stresses. The exact values are obtained simply from solutions of the discretized problem. Computing the values "before correction" we assume  $\delta_1 = 0$ ,  $\delta_2 = 0$  and take into account only the first integrals over the circles  $\Gamma_R(\mathbf{x})$  with  $R = 0.05$ . The corrected values contain nonzero  $\delta_1$ ,  $\delta_2$ . As we see from Fig. 4–6 the corrections are an order of magnitude smaller than the real values, but are important in the regions of higher stresses. As a byproduct we have obtained the formula for computing stresses at a point, which is more laborious than simple differentiation of the discrete solution, but also more accurate (not discussed in detail here).

### 3.6 Testing the modified energy

We shall check numerically, how the modification of the elastic energy allows us to approximate the influence of the small hole inside the domain. To this end we consider the same domain as in the last section with the same external loading. In this domain we shall make the hole at the point  $\mathbf{x}_0 = [0.5, 0.7]$  of the radius  $\rho = 0.05$ . The surrounding ring  $\Gamma_R$  will have radius  $R = 2\rho$ . In order to make comparisons, we shall need the "exact" solution. As

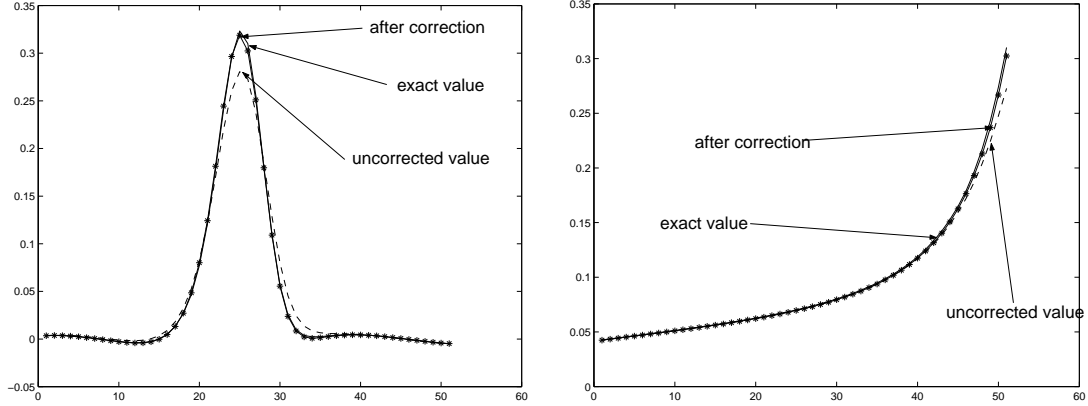


Figure 5: Graphs of the  $\epsilon_{22}$  stress component along the horizontal line near the upper edge (left) and the vertical line in the middle.

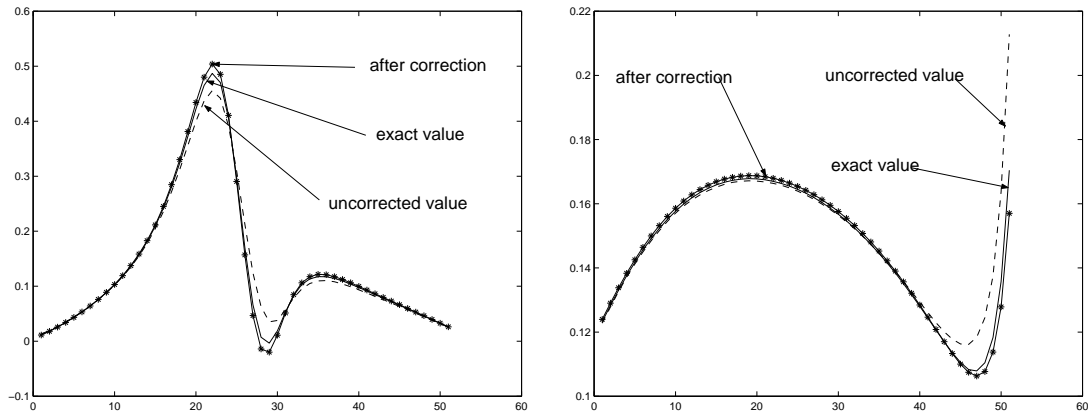


Figure 6: Graphs of the  $\epsilon_{12}$  stress component along the horizontal line near the upper edge (left) and the vertical line in the middle.

an approximation of such a solution up to the terms of the third order in  $\rho$  will serve us the solution of the inhomogeneous problem with the singular right-hand side (force term) depending on the solution in the intact domain  $\mathbf{u}_0$ :

$$\mathbf{f} = -2\pi\rho^2 \begin{bmatrix} \epsilon(\phi_1) : \sigma(\mathbf{u}_0) \\ \epsilon(\phi_2) : \sigma(\mathbf{u}_0) \end{bmatrix}$$



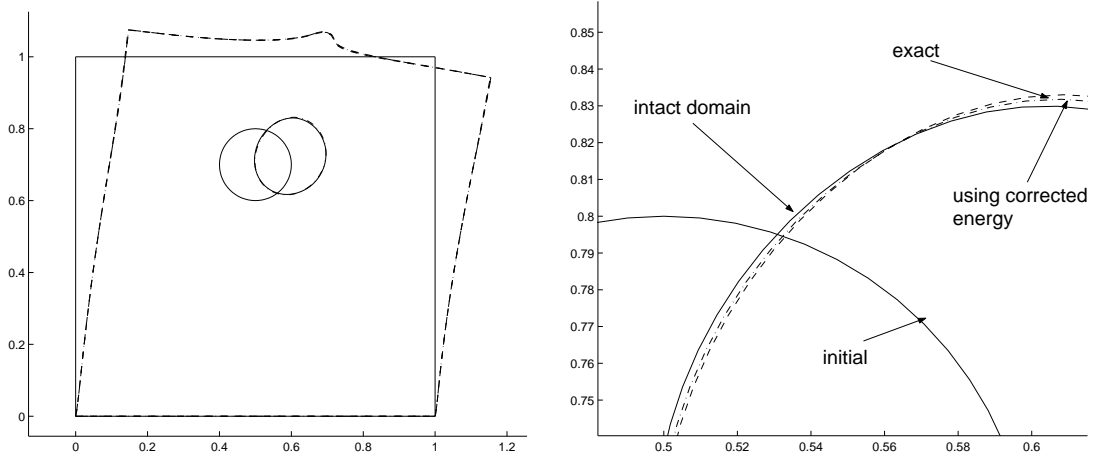


Figure 7: On the left the domain in initial and deformed states (exaggerated). The positions of the ring  $\Gamma_R$  computed in different ways nearly indistinguishable due to scale. On the right the blow-up of the part of the ring.

where  $\phi_1$  and  $\phi_2$  are the column vectors of the form

$$\phi_1 = \begin{bmatrix} \delta(\mathbf{x} - \mathbf{x}_0) \\ 0 \end{bmatrix}, \quad \phi_2 = \begin{bmatrix} 0 \\ \delta(\mathbf{x} - \mathbf{x}_0) \end{bmatrix}.$$

The validity of such an approach is proved in [24] and has been verified numerically in the sections concerning Laplace equation. In the weak formulation the term involving  $\mathbf{f}$  reduces to the point value of  $-2\pi\rho^2 \epsilon(\mathbf{u}) : \sigma(\mathbf{u}_0)$ .

First we shall compare the values of displacement along the boundary of the ring, where the errors should be the greatest.

As we see in figures 7,8, the corrected energy gives results being in a very good agreement with the "exact" ones.

Next we shall investigate the influence of an additional term containing  $\delta_1$  and  $\delta_2$  in the modification of energy, see (100). As "simple" modification we shall call the one assuming  $\delta_1 = \delta_2 = 0$ . The figures 9 and 10 show the values of *relative* corrections to the displacement, i.e. divided by  $\max[|u_i|]$  computed along the appropriate line. One can see that this influence is small, in practice negligible. Nevertheless, the additional term is needed for the asymptotic correctness of the formulae.

Let us stress here that the values of  $\epsilon$  and  $\sigma$  appearing in the principal part of the energy correction depending on  $e_u(0)$  do contain  $\delta_1$  and  $\delta_2$  in agreement with (75). Smaller by two orders of magnitude turned out the influence of energy correction term in which  $\delta_1$  and  $\delta_2$  appear alone as multiplicative factors.

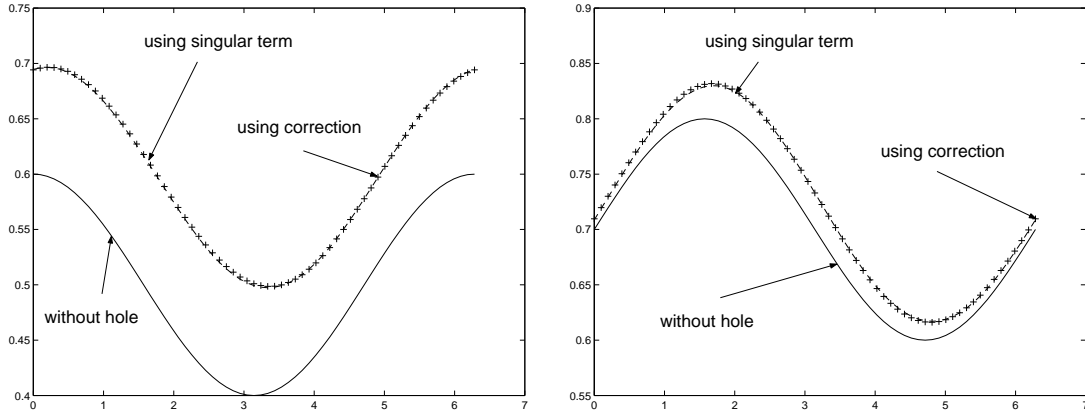


Figure 8: The values of  $u_1$  (left) and  $u_2$  (right) components of the displacement along the ring  $\Gamma_R$  computed in different ways.

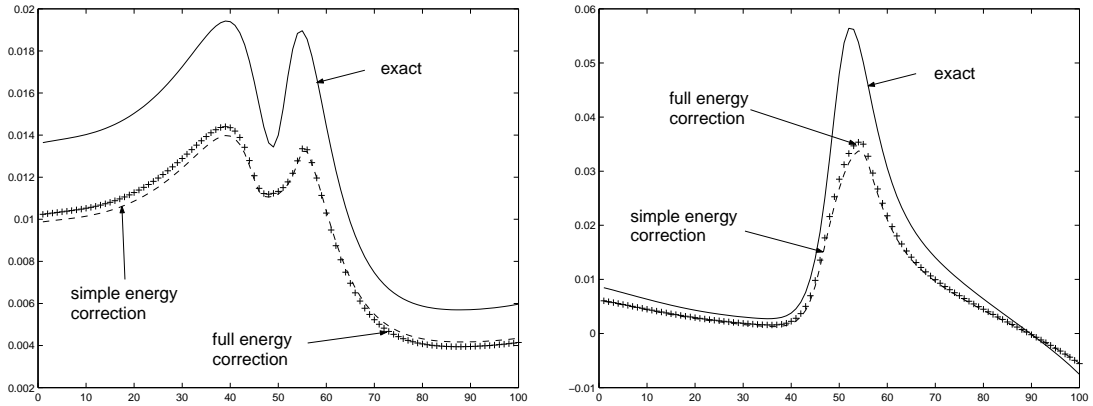


Figure 9: The relative corrections to the displacements  $u_1$  (left) and  $u_2$  (right) along the line  $x_2 = 0.8$ , which touches the ring  $\Gamma_R$ .

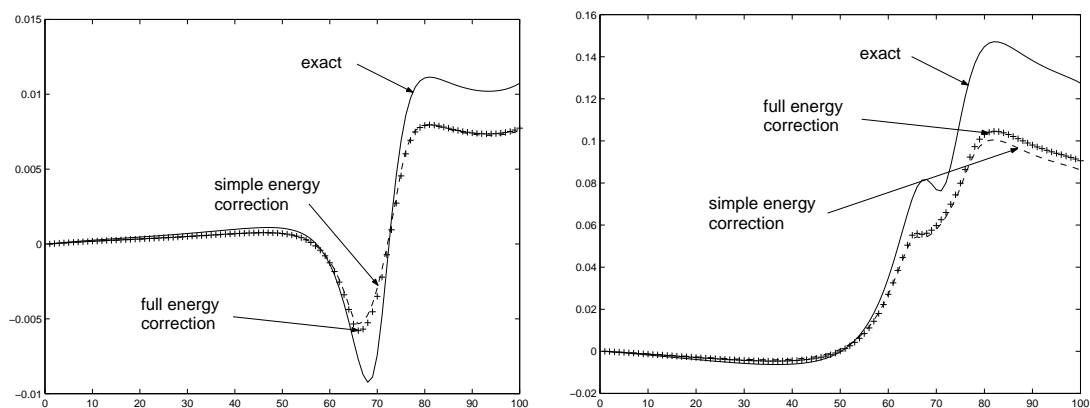


Figure 10: The relative corrections to the displacements  $u_1$  (left) and  $u_2$  (right) along the line  $x_1 = 0.6$ , which touches the ring  $\Gamma_R$ .

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